Random Network Models (Luke, 2015)

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The Role of Network Models

In this chapter, a number of basic mathematical models of network structure and formation are covered.

1. Serve as baseline or comparison models for empirical social networks.
2. Act as building blocks for more complex network simulations.

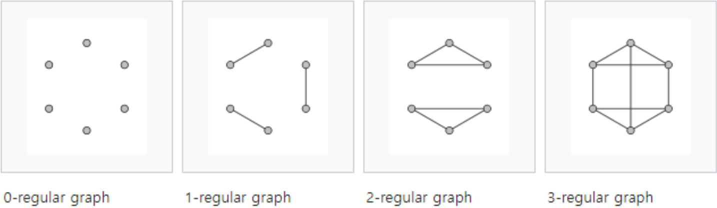
These are important models in the history of network science, but they are still useful today to provide insight into fundamental properties of social networks.

Metric

- transitivity (clustering coefficient)
  Transitivity is an index indicating how well nodes connected to a node i. It bounds between 0 and 1.
- diameter
  Diameter is an indicator for determining the size of a network, the longest distance between nodes in the network. (→ Average path length)

Regular Network model

Regular networks Definition: Each node has exactly the same number of links(edges).



Erdős–Rényi Random graph model

Erdős–Rényi model is the first developed random graph model.

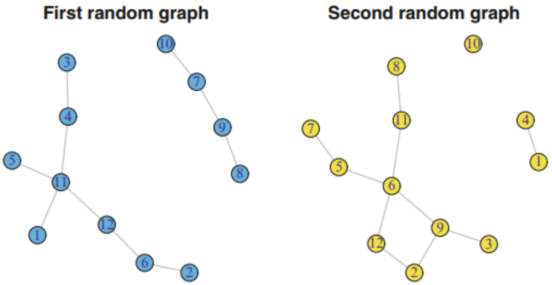
$G(n, m)$ or $G(n, p)$

- G: random graph
- n: nodes
- m: edges
- p: probability

A random graph is produced by specifying the size of the desired network, edges, or the probability of observing an edge.

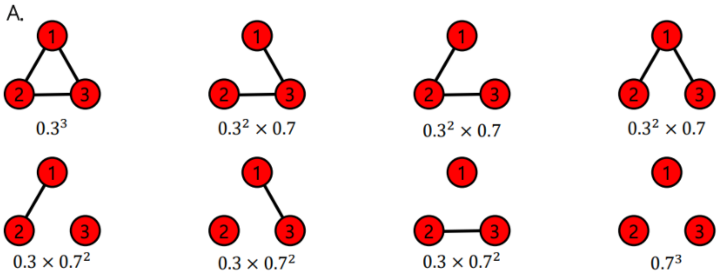
Erdős–Rényi Random graph model(Cont'd)

Let's calculate the probability for each graph, G(10,10).



Erdős–Rényi Random graph model(Cont'd)

Let's calculate the probability for each graph, G(3,0.3).



Erdős–Rényi Random graph model(Cont'd)

- For the G(n,m) model
  - The number of nodes is fixed
  - Average degree for a node: 2m/n
  - However, other properties are hard to analytically obtain
- We will use a slightly different model, G(n,p)
  - The number of nodes is fixed
  - Furthermore, we fix probability p, that every possible edge between the n nodes appears in the graph
  - Note: the number of edges in this network is **not** fixed

* average degree= mean degree = mean node degree = average degree of a node

Erdős–Rényi Random graph model(Cont'd)

Discoveries

- The average degree(c) is related to graph size(n) and edge probability(p).

$$c = (n-1)p$$

- For large n, the network will have a Poisson degree distribution.

Erdős–Rényi Random graph model(Cont'd)

Proof) Probability that we get G(n,m)

$$P(G(n,m)) = \binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m}$$

Average number of edges for a network model: $\binom{n}{2}p$

Erdős–Rényi Random graph model(Cont'd)

Proof)

$$\begin{aligned}\text{Average degree(c)} &= \sum_{m=1}^{n} \frac{2m}{n} p(G(n,m)) \\ &= \frac{2}{n} E(G(n,m)) \\ &= \frac{2}{n}\binom{n}{2} p = \frac{2}{n}\frac{n!}{2!(n-2)!} p \\ &= (n-1)p.\end{aligned}$$

c.f., Average degree(mean node degree) for G(n,m) model is 2m/n, 8pg.

Erdős–Rényi Random graph model(Cont'd)

An edge in a random graph, it's connected with equal probability p with each N-1 other nodes. Hence p_k that it has degree exactly k is given by the binomial distribution (Newman et al., 2002).

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

We know average degree is c, then rewrite

$$p_k = \binom{n-1}{k} (\frac{c}{n-1-c})^k (1 - \frac{c}{n-1})^{n-1} \cong \frac{c^k}{k!} e^{-c} \cong Pois(c)$$

Erdős–Rényi Random graph model(Cont'd)

Discoveries (Cont'd)

- Each nodes does not have to be connected to too many other members for the network itself to be connected.

$$c = (n-1)p \leftrightarrow p = \frac{c}{n-1}, \text{as n increase}, p \rightarrow 0.$$

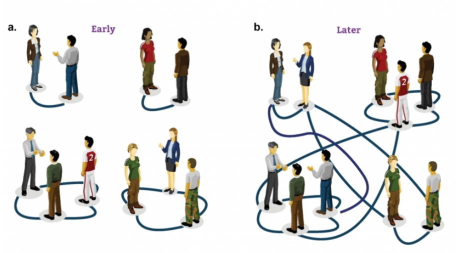
- It derives small transitivity.

Small world model

However, Random graph model(low clustering) does not describe the properties of many real-world social networks(high clustering).

Small world model (Watts and Strogatz, 1998) networks have more realistic levels of transitivity. (e.g., six degrees of Kevin Bacon in facebook.)

Small world model



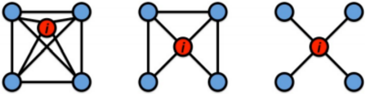
Watts and Strogatz motivated by this example.

Local Transitivity

Local Transitivity

$$C_i = \frac{2e_i}{k_i(k_i - 1)}$$

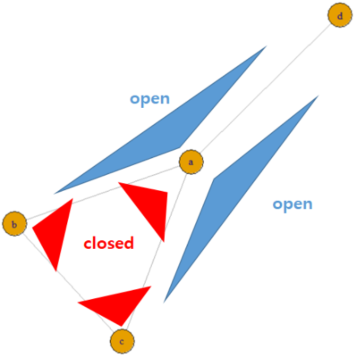
where e_i is the number of edges between the neighbors of node i



$C_i = 1$ $C_i = 1/2$ $C_i = 0$

In the middle plot, connections between 4 nodes are 4*3 = 12, and interconnected edges are 3, transitivity = 3*2/12 = 1/2.

Global Transitivity

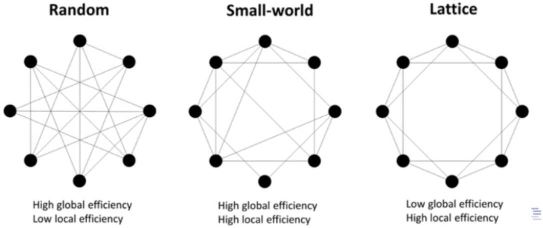


$$\text{Global Transitivity} = \frac{3 * \text{number of triangles}}{\text{Number of 3 nodes connected}}$$

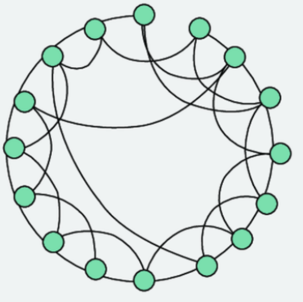
Small world model

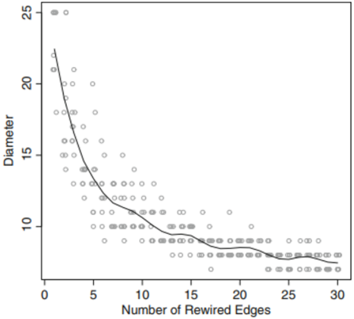
Small-world networks tend to contain cliques, and near-cliques, meaning sub-networks which have connections between almost any two nodes within them. This follows from the defining property of a high transitivity.

Even if the entire network is large, the entire network can be closely connected by some specific nodes



Small world model





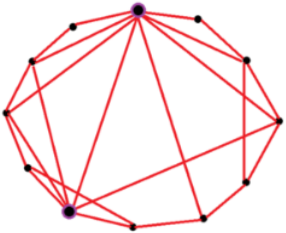
Figures show how quickly rewiring reduces the diameter of a network in the small-world model.

Small world model

L: distance between two randomly chosen nodes

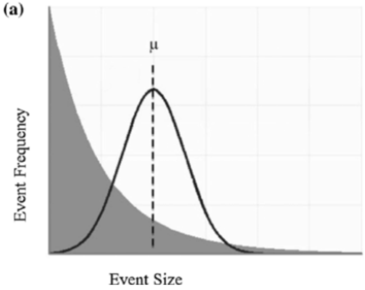
N: Number of nodes in the network

$$L \propto logN$$



Scale-Free Model

Previous two mathematical network models are not representative of many real-world social networks. Observed networks have heavy-tailed degree distribution (power law).



Scale-Free Model

e.g., Some websites have a very large number of other websites connected to them, but most websites have only a few connections.

We can call this situation as *cummulative advantage*, or *preferential attachment*.

Scale-Free Model

Scale: Randomly choose one node, On average, if a node of degree is obtained stably, it is called the scale of the network. scale-free network distribution (Power Law):

$$p(k) = Ck^{-\gamma}, \quad k : degree$$

Variance is ∞, therefore scale-free.

Scale-Free Model

$E(k)$: average of degree.

In the scale-free condition, $E(k)$ cannot represent k.

$$E(k^2) >= \int_{k_{min}}^{k_{max}} k^2 p(k) dk$$
$$= \int_{k_{min}}^{k_{max}} k^2 C k^{-\gamma} dk$$
$$= C \frac{{k_{max}}^{3-\gamma} - {k_{min}}^{3-\gamma}}{3-\gamma}$$

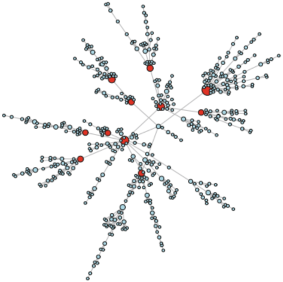
For large n(node), $k_{max} \rightarrow \infty$ and $k_{min} = Constant$.

$\therefore \quad E(k^2) \quad \rightarrow \quad \infty$, when $2 < \gamma \leq 3$

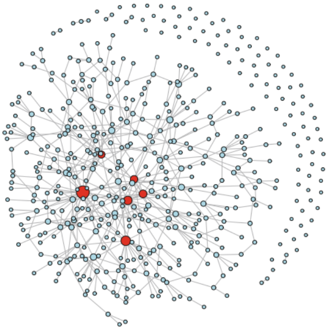
Scale-Free Model

'Rich-gets-richer' phenomena has been shown to lead to the power-law distribution in networks.

It's more complicated, because this is a network growth model not static structure model.Also there are *hubs* with degree > 9.

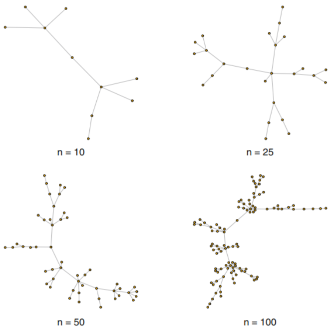


Scale-Free Model



With modified options, there are some isolated nodes and it's more realistic.

Scale-Free Model



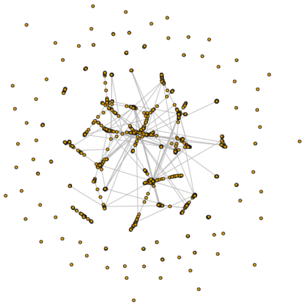
Growth of networks using preferential attachment model.

Empirical Networks

Data: Communication ties among 1,283 leaders of local public health departments. (lhd data)

Builds three network models that have the same size and approximately the same density as the lhd network.

Empirical Networks



Lhd network has much higher transitivity than any of the models.

Name	Size	Density	Avg. degree	Transitivity	Isolates
Erdos-Renyi	1283	0.003	4.404	0.002	21
Small world	1283	0.003	4.000	0.088	1
Preferential attachment	1283	0.002	2.195	0.003	109
Health department	1283	0.003	4.221	0.306	58

Luke, D. A. (2015). *A user's guide to network analysis in r* (Vol. 72). Springer.

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